AN INTERACTIVE GOAL PROGRAMMING APPROACH

TO MULTI-ITEM INVENTORY SYSTEMS

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SUMMARY

This thesis is concerned with the devopment of an interactive goal programming approach to a class of stochastic multi-objective inventory problems. It centers on the assumption that there exist multiple conflicting objectives which cannot be incorporated to form one single objective function to be optimized. This normally occurs when there are one or more cost parameters which are not readily quantifiable.

In the conventional approach to optimization, management participates only in estimating cost parameters and approving or disapproving the solution obtained by the operations research practitioner. In contrast, the interactive approach proposed here allows the decision maker to systematically progress to a "most preferred" solution. Thus the decision maker takes an active part in the development of a solution which is more indicative of what the decision maker sees as desirable tradeoffs among the multiple objectives.

To obtain each new point in the feasible solution space a Lagrange multiplier approach is used. The basis of the interactive nature of the algorithm is dictated by the mathematical formulation of the problem.

The algorithm is coded in Fortran and a numerical example is presented. The example is used to illustrate the interactive nature of the approach.

CHAPTER I

INTRODUCTION

The classical theory of inventories incorporates models which deal with many different inventory situations under a wide variety of assumptions. Most of that body of knowledge operates under the assumption that cost factors such as ordering costs, carrying costs and stockout costs are known. The general approach of this classical theory is to construct a total cost function from these costs, and to proceed to optimization of the function over a given set of variables.

There exists an inherent problem with the classical approach. In trying to apply the theory, practitioners have found that very often one or more of the needed cost factors cannot be found. This is particularly the case with stockout costs.

Ordering costs are, more often than not, readily measurable from accounting records. Holding costs, although harder to determine, can be arrived at by careful examination of accounting records and through estimation by management. In any case, most companies have figures for ordering and holding costs which they feel are good estimates of the true costs. On the other hand, stockout costs tend to be rather elusive. They are the most difficult to quantify of the three cost factors mentioned.

When any of the cost factors are not available, or perhaps nearly unquantifiable, the classical theory of inventories cannot be used, directly, to solve any given problem. When this happens, and one cannot construct a representative total cost function, but rather there exist distinct and non-additive parts, it is said to be a problem with multiple objectives. It is the problem of optimizing over multiple objectives that is the subject of this thesis.

Since stockout costs are most often unquantifiable (whereas the others are more easily obtained), therefore, this research centers around the assumption that ordering costs and carrying costs are given. Stockout costs, then, remain as the reason for having to use multi-objective techniques.

Description of the Problem

As previously mentioned, the objective in any inventory problem is to minimize total cost. The fact that stockout costs are unobtainable makes it impossible to measure the objective in common units (dollars). This is equivalent to splitting the total cost function into two parts. One part can be represented in dollars (ordering costs plus carrying costs) while the other is left as some function of the number of stockouts incurred by the system.

Although the dollar value of each item's stockout penalty is not known, certain things are known about these penalties. For example, one very basic fact is that stockout penalties differ between the different inventoried items. Furthermore some items are usually known to have higher stockout penalty costs than others. It thus makes no sense, in a multi-item inventory system, to say that 200 stockouts per year are better than 300. The comparison can only be made after gaining information as to how the overall system is performing in reference to other criteria or measures of performance.

Since a simple objective, such as total stockouts per year, cannot be used independently to ascertain the "goodness" of any solution, other criteria must also be considered as objectives. The objectives that will be used in selection of alternatives are those which describe the attributes of a multi-item inventory system. These criteria or objectives will be defined later.

The problem of optimizing over multiple objectives has been described by Geoffrion et al. [9] as

$$\text{Extremize} \quad U[f_1(x), f_2(x), \ldots, f_n(x)]$$

$$\text{Subject to} \quad x \varepsilon \chi$$

where x is the vector of decision variables, U is the decision makers preference function over the n objectives and X is the set of feasible solutions.

The function U is not explicitly known. If it were, there would be no need to use a multi-objective approach. U, then, can be thought of as a function which gives preferences to combinations of the n objectives.

One further point must be made about the multi-objective approach to optimization. In referring to extremizing over the set of objectives it must be understood that the "optimum" reached by any particular decision maker is not an absolute optimum for the system. The reason for this is that the point, chosen as the best by that decision maker, is a function of how he perceives tradeoffs between the objectives. That is to say that, more than likely, the functions U are not the same for any set of decision makers. Thus different decision makers may consider two different points as "optimum" respectively. This fact is not inconsistent with common business practices since in any enterprise decisions are made as to investment, mergers, sales, etc., on these same basis. Along this line, Roy [17] states "Such a pragmatic approach may appear as nonscientific, accustomed as we are to solve a problem only after it has been completely and clearly formulated. Here we begin to solve a problem when the nature of 'best choice' we are looking for is still fuzzy, and exploit the work done step by step, in progressing to a definition of the unknown. I see nothing irrational or clumsy in this approach. On the contrary, I believe it is very close to what is effectively done by the decision maker himself."

Purpose of the Research

Multiple objective systems occur frequently in practically every branch of operations research and engineering. As previously mentioned inventory systems tend to be in the category of multiple objective systems. The purpose of this research is to demonstrate the applicability of the concept of multi-objective optimization to multi-item inventory systems. A further objective of this study is to develop and illustrate a method for "optimizing" multi-item inventory systems using the philosophy of multiple objective optimization.

Scope of the Research

In dealing with multi-item inventory systems one can encounter a wide variety of properties among the items. Some properties dictate the type of model which appropriately describes the behavior of the items. Therefore in a typical inventory situation many models are needed to fully and accurately represent the system. This variety tends to complicate the solution or optimization procedure tremendously, even if all of the relevant cost factors are known.

With no loss of generality, a system of a more homogenous nature can be considered. For the purpose of illustration of certain principles, it is appropriate to consider a system where demand distributions are similar and can be thought of as continuous, all items are stocked under the same operating doctrine, etc. More complex system containing all mixtures of dissimilar items can be thought of as a combination of simpler, more homogenous systems.

This research is not aimed at finding a generalized solution for all inventory problems. Rather the emphasis will be on laying out a procedure for solution of multi-objective problems with special application to inventory systems.

The particular inventory model used for illustration of the solution procedure, has been chosen because of its wide-spread use in the standard literature on inventory theory. Furthermore it has several qualities which make it desirable to work with in this framework. Nevertheless the techniques developed in this research are valid and useful in dealing with all the inventory models currently in the literature.

Method of Procedure

The first section of this thesis is devoted to reporting a survey of published literature which is relevant to multi-objective and interactive optimization procedures. This section establishes the background necessary for an under-standing of the problem area being investigated. The basic formulations of multi-objective models are presented along with descriptions of interactive optimization procedures. Current and proposed applications of these principles are also discussed.

The next section describes and develops a new solution procedure for multi-objective problems based on the theory of Lagrange multipliers. A problem formulation and model are developed along with a general algorithm to obtain a solution. The algorithm is of the "man-machine/interactive" type and has several similarities to some which exist presently in the literature.

The following sections are devoted to the development of a specific inventory model which can be solved using a "multi-objective/interactive" approach. The model is developed first using the classical optimization (single objective function) approach; then it is used to obtain multiple objectives and criteria over which an optimization can be performed.

A two goal example of the "multiple objective/interactive" solution procedure is given using the previously developed inventory model. The basic features of the proposed algorithm are demonstrated by means of a computer program. Samples of the program output are shown and explained. The extension of the approach to a four goal environment is then made.

The final section of this thesis presents a summary of the results established in the previous sections and draws conclusions from the research effort. Comparisons are made between the proposed solution method and those which were considered in the literature. Also, areas are given into which further research effort may be directed.

CHAPTER II

MULTI-OBJECTIVE AND INTERACTIVE OPTIMIZATION

The subjects of multi-objective and interactive optimization are relatively new fields in Operations Research. Aside from some work on goal programming which took place prior to 1965 the majority of the literature on these fields has been written after 1969.

Most contributors to the literature of Operations Research have concentrated on problem formulation, taxonomy and algorithmic development. For the most part the problem formulations were of the "one-objective under a set of constraints" type and the algorithms were such that a computer could perform the optimization. However it is important to consider the many problems and situations which demand that multiple objectives and/or man machine interactions be used in arriving at a satisfactory solution.

The following sections will summarize several approaches used to deal with multi-objective optimization and man-machine interactive optimization. First, goal programming will be explored as a means of "optimizing" over multiple criteria. Next several interactive programming procedures will be considered. All but vector maximal decomposition deal with linear objective functions. Lastly a method called interactive goal programming will be described. This method, as the name suggests, uses the ideas of goal programming but puts them in an interactive format which is generalized to include non-linear objective functions.

Goal Programming

To deal with the problem of "goal attainment" Charnes and Cooper [4] proposed the method of goal programming (GP). It was used to deal with multiple linear objectives and linear constraints of the form

$$a_i x = b_i$$

where b_i is the ith goal target and a_i is the ith row of technological coefficients. Thus any deviation above or below the target b_i can be represented by y_i^+ or y_i^-, respectively, so that the individual goals can be expressed as

$$a_i x - y_i^+ + y_i^- = b_i$$

Y. Ijiri in his dissertation and later in [12] refined and extended the techniques of GP. He formulated the GP problem as

$$\begin{aligned} &\text{Min} \quad 1 \cdot y_i^+ + 1 \cdot y_i^- \\ &\text{s.t. } Ax - Iy^+ + Iy^- = b \end{aligned} \qquad \text{(II-1)}$$

where 1 is a vector with all elements equal to 1 and I is the identity matrix. Of course if (II-1) has a solution $x \geq o$, $y^{+}=o$, $y^{-}=o$ then all the goal targets b_i are said to be simultaneously compatible. This is usually not the case in real business problems.

To deal with incompatible multiple goals Ijiri proposed an ordering and weighting scheme which would allow the decision maker to set goal priorities and importance weighting [12, p. 45].

Most recently GP has been extended and popularized by people like S. M. Lee and T. W. Ruefli. Their contributions are mainly of an applied nature with emphasis on decision analysis and decentralized organizations [16,18,19].

Interactive Programming

The term interactive programming is given to any optimization procedure which uses some algorithm or heuristic coupled with a human decision maker to reach an "optimum" or "best" solution to a particular problem. This coupling or interaction with a human element becomes necessary or desirable in many optimization problems.

The need for interaction can emerge from one of several factors. Primarily an interactive mode can overcome the barrier or incomplete information on the objective function. Other factors which call for the use of Interactive Programming are the inability of many algorithms to obtain "good" solutions to large scale problems and the ability to exploit unreproducable human problem solving capabilities.

For example the work of Krolak et al. [13,14] is based on the theory that a heuristic to solve large scale network problems can be greatly improved if at each step humans are allowed to aid in the optimization process.

The approach used by Krolak [13] is as follows: (1) man defines the problem, (2) the computer organizes the data in a fashion that isolates the problem's features and gives several solutions to show how the data might be organized into a whole by various methods, (3) man attempts to organize the data into a solution, (4) man makes a comparison between the computer solutions, his solution and the computer's organization of the data and creates a composite solution, (5) with the composite solution, man uses the computer to review and investigate various local and regional problems isolated by Step 4, and (6) using his judgement and whatever theoretical information might be at hand, man continues Step 4 and 5 until he has either exhausted all of the potential benefits to be derived or until further effort will be only marginally beneficial.

The above procedure can serve to make the decision maker aware of how the various relationship in the problem interact and contraint his solution. This is an educational process which allows him to adopt to future changes more intelligently. In addition this process gives the decision maker more insight into the nature of his problem.

Baker et al. [1] applied the interactive philosophy to a multi-level hierarchical organization. Here the computer gives an initial solution then through several steps of questions and answer a new solution and/or alternative plans of action can be developed by the decision maker.

Still within the linear programming framework, Benayoun, Tergny and Keuneman [2] introduced the concept of "best compromise" to replace that of best solution (in the sense of optimum). Their approach is named POP (Progressive Orientation Procedure) and is used to optimize over multiple goals (in the same context as goal programming). In [2] they study three cases according to the information available on the relative importance of the objective functions. The three cases are (1) it is quantifiable, (2) it is known but not quantifiable, and (3) it is completely implicit. Their algorithm, named STEM, is an interactive mechanism which allows the integration of the decision maker's response to very simple questions for guiding the exploration of the feasible set and finding the best compromise.

<u>Multiple Non-Linear Objectives</u>

In 1970, A. M. Geoffrion introduced Vector Maximal Decomposition Programming (VMDP) [8]. The philosophy of VMDP is much like that of the P.O.P. method discussed above. It optimizes over multiple criteria with an implicitly defined preference function. But there are two differences in the procedures: (1) VMDP handles non-linear objective functions, and (2) the interaction used is dictated by the nature of the mathematical programming algorithm itself.

The approach to VMP suggested by Geoffrion [8] falls in the man-machine interactive category. It is based on standard ascent methods that would be applicable to VMP if the preference function were explicitly known. Geoffrion states, "the appeal of such an approach seems so obvious that one would expect it to have been thoroughly studied previously." However he is able to cite only one other independent work by Boyd [3] as being along the same lines as that suggested in [8].

An interesting application of Geoffrion's ideas can be found in [9]. Here Geoffrion, Dyer and Feinberg have applied the VMDP approach to the operation of an academic department. They show that although the overall preference function was not known explicitly their algorithm required only such local information about the preference function as was actually needed to carry out the optimization.

Geoffrion, Dyer and Feinberg put the VMPD approach as follows: "adopt a mathematical programming technique of known efficiency, but implement it so as to require minimal information from the decision maker concerning his preferences over the criteria." In [9] the mathematical technique used was the Frank-Wolfe algorithm for non-linear programming.

This algorithm was chosen because of its simplicity and its appropriate theoretical properties.

The multi-criterion problem, as stated in [9], is:

$$\text{Maximize} \quad U[f_1(x), f_2(x), \ldots, f_n(x)] \tag{II-2}$$

$$\text{Subject to} \quad x \varepsilon \chi$$

where $f_1, \ldots, f_n$ are n distinct criterion functions of the decision vector x, χ is the constrained set of feasible decisions and U is the decision makers preference function over the n criteria.

The Frank-Wolfe algorithm applied to the above problem is given as follows:

Step 0 Choose an initial point $x_1 \varepsilon \chi$. Set k=1.

Step 1 Determine an optimal solution y_k of the direction-finding problem

$$\underset{y \varepsilon x}{\text{Maximize}} \quad \nabla_x \, U[f_1(x_k), \ldots, f_n(x_k)] \cdot y \tag{II-3}$$

Let $d_k = y_k - x_k$

Step 2 Determine an optimal solution of the step-size problem

$$\underset{o \le t \le 1}{\text{Maximize}} \quad U[f_1(x_k + td_k), \ldots f_n(x_k + td_k)] \tag{II-4}$$

Let $x_{k+1} = x_k + td_k$, $k = k+1$ and return to Step 1

For (II-2) to be a multi-criterion problem U cannot be explicitly known. Therefore neither Step 1 nor Step 2 can be carried out entirely by a computer. This makes it necessary to interact with the decision maker in order to obtain certain information which will allow Steps 1 and 2 to be performed.

The Frank-Wolfe algorithm is rendered interactive by Geoffrion, Dyer and Feinberg in [9]. However a brief synopsis of their work will be helpful in clarifying the nature of the interactive approach.

When applied to (II-3), the chain rule yields

$$\nabla_x U[f_1(x_k),\ldots,f_n(x_k)] = \sum_{i=1}^{n} \left(\frac{\partial U}{\partial f_i}\right)^k \nabla_x f_i(x_k) \qquad \text{(II-5)}$$

where $(\partial U/\partial f_i)^k$ is the ith partial derivative of U evaluated at x_k and $\nabla_x f_i(x_k)$ is the gradient of f_i evaluated at x_k. Notice that the optimal solution y_k of (III-2) is not affected by positive scaling of the objective function. Since the $(\partial U/\partial f_i)^k$, although unknown, are constants it is possible to divide through by any positive coefficient $(\partial U/\partial f_i)^k$ without changing the problem. The criterion whose coefficient is used for this purpose is called the reference criterion. Hence

$$\underset{y\varepsilon X}{\text{Maximize}} \sum_{i=1}^{n} \left(\frac{\partial U}{\partial f_i}\right)^k \nabla_x f_i(x_k) \cdot y = \underset{y\varepsilon X}{\text{Maximize}} \sum_{i=1}^{n} w_i{}^k \nabla_x f_i(x_k) \cdot y$$

where

$$\begin{aligned} w_i{}^k &\equiv (\partial U/\partial f_i)^k/(\partial U/\partial f_1)^k, \quad i = 1,2,\ldots n \\ &= (\partial f_1/\partial f_i)^k \qquad\qquad\quad, \quad i = 1,2,\ldots n \end{aligned} \tag{II-6}$$

Hence one way to determine the $w_i{}^k$ is to find that change in the reference criterion "exactly compensates" for a change in the ith criterion, with all other criterion remaining constant. It is possible to use a criterion with a negative coefficient as the reference criterion, in which case one would divide through by the negative of the coefficient.

All the information necessary to carry out step 1 is contained in the $w_i{}^k$. On the other hand, Step 2 must be accomplished by the decision maker directly. It is up to the decision maker to decide upon a step size and communicate this information to the computer before a new iteration can be performed.

The decision on step size can be made with the help of the interactive program. As proposed by Geoffrion, Dyer, and Feinberg [9, pp. 7-8] this can be accomplished by plotting each objective $f_i(x_k)$ as a function of a single variable t and allowing the decision maker to choose a point $o \le t \le 1$ which in his opinion achieves maximum increase in U.

Further exploration into the use of the interactive Frank-Wolfe algorithm was conducted by Dyer [5] at the Western Management Science Institute, U.C.L.A. In this study nine students were asked to solve a hypothetical problem, involving multiple criteria, using basically the same method used in [9]. Further the subjects were asked to rate the procedure in terms of the difficulty of its use and their confidence in the solution obtained. The analysis of their experiences indicated that a time-sharing program based on Geoffrion's approach can be used successfully by relatively unsophisticated decision makers.

Interactive Goal Programming

As suggested by the name, interactive goal programming (IGP) is a combination of the goal programming concepts with the interactive capabilities. Also IGP is an extension of the goal programming formulation into a non-linear environment. In a sense, interactive goal programming was introduced by Dyer [6] to provide a "bridge" between goal programming and the vector maximal decomposition programming strategies proposed by Geoffrion.

In his formulation of the goal programming problem Dyer replaces the linear form $a_i x$ with a general function $f_i(x)$.

Thus the general GP problem becomes

$$\text{Min} \sum_{i=1}^{n} w_i y_i^-$$

$$\text{s.t. } f_i(x) - y_i^+ + y_i^- = b_i \text{ for } i=1,\ldots,n$$

$$y_i^+, y_i^- \geqq 0$$

where the w_i are measures of relative importance which are distributed to each criterion, and y_i^+ and y_i^- are respectively positive and negative deviations of $f_i(x)$ from b_i.

Dyer noted that the formulation of the "one-sided" goal programming problem is similar to the formulation of the sub-problem of the Frank-Wolfe algorithm. Consequently he adopted a solution procedure similar to that of Vector Maximal Decomposition Programming. The procedure is as follows:

Step 1. Determine the goal or objectives b_i. Each b_i must be chosen such that obtaining a value of $f_i(x)$ higher than b_i is of no value to the decision maker.

Step 2. Determine an initial feasible point x^k. Calculate $f_i(x^k)$. Let k=o.

Step 3. Determine w_i by interacting with the decision maker as described in [4]. The w_i are defined by equation (II-6).

Step 4. Solve the "one-sided" goal programming problem.

$$\text{Min} \sum_{i=1}^{n} w_i^k y_i^-$$

$$\text{s.t. } f_i(z)+y_i^- \geqq b_i \text{ for } i = 1,\dots,n$$

$$y \geqq o,$$

for the optimal values y_i^{k-} and the associated z^k. Let $d^k = z^k - x^k$.

Step 5. By interacting with the decision maker, determine an approximation to the step size $t^k (o \leq t \leq 1)$ which maximizes $U(f(x^k+td^k))$.

Step 6. If $U(f(x^k+t^kd^k)) \leq U(f(x^k))$ terminates the procedure. Else set $x^{k+1} = x^k + t^kd^k$, find $f(x^{k+1})$ increase k by one and go to Step 3.

The draw back of Dyer's algorithm is that at each iteration one must solve the "one-sided" GP problem of Step 4. Depending on the form of the $f_i(z)$ this task may turn out to be quite difficult in itself.

It is important to note that in most interactive algorithms, especially those of Geoffrion and Dyer, the decision maker is solving a problem in criteria space. That is, the decision maker is never confronted with values of the decision variables x. In fact, it is not necessary for him to understand the interactions in the decision variables to make his decisions.

Proposed Areas of Application

Whenever there exists a problem whose objective function is not well defined, the multi-objective and interactive optimization approaches are of great use. In many cases these approaches help to apply quantitative techniques to problems previously thought be be unquantifiable.

Roy [17] states, "In many real problems (choice of investments for a firm or a community; selection, assignment or remuneration policies; product planning, scheduling or sequencing problems...) comparison between alternatives must be made on the basis of multiple heterogeneous and complex consequences (dealing with cash-flow, shares, market condition, future investment possibilities, quality, comfort, security, growth, welfare,...)." So it can be easily seen that there exist extensive needs for this type of solution procedure.

Other uses for multi-objective and interactive optimization have also been studied. Among these are:

(1) solution to managerial accounting problems (Ijiri [12])

(2) Multi-level (hierarchical) organizations (Baker [1], Geoffrion [8], Ruefli [19])

(3) Resource allocation in academic departments (Geoffrion, Dyer and Feinberg [9])

(4) Solution of the traveling salesman problem (Krolak, Felts and Marble [13])

(5) Quality control optimization (Hindelang [11])

Although this list is not exhaustive it gives an idea of the diversity of situations in which multi-objective and interactive optimization techniques can be used.

CHAPTER III

INTERACTIVE GOAL PROGRAMMING: A LAGRANGE MULTIPLIER APPROACH

In the previous chapter several approaches for dealing with the problem of multiple criteria, and unknown objective function, were considered. Surprisingly enough, the generalized Lagrange multiplier technique has not been employed to obtain an algorithm for the solution to this problem. It is surprising indeed since the use of the Lagrangean technique on the general goal programming problem exhibits extremely nice properties. These properties become even more evident when results obtained from the Lagrangean formulation of the general GP problem are considered in terms of interactive decision models.

The following sections are concerned with the development of an algorithm for the multi-criteria problem using the generalized Lagrange multiplier technique. The algorithm will be such that it can be rendered interactive in the same sense as that of Geoffrion, Dyer and Feinberg [9]. That is to say that the interaction will be dictated by the nature of the mathematical algorithm and not superimposed ad hoc.

The Lagrange multiplier approach will yield a less general algorithm than that proposed by Dyer in [6], however the algorithm proposed here can exploit certain special properties which are exhibited by some mathematical models.

Lagrangean Formulation of the Goal Programming Problem

Consider a problem with n goals. In general each goal will be represented by a non-linear function $f_i(x)$, where x is an m dimensional vector of decision variables. Each goal will be assigned a target value b_i and any deviation of $f_i(x)$ from b_i is given by a variable y_i which is unrestricted in sign.

If w_i is the non-zero weight or penalty attached to each unit of deviation y_i the GP problem is

$$\text{Minimize} \quad R = \sum_{i=1}^{n} -w_i y_i \tag{III-1}$$

$$\text{Subject to} \quad f_i(x) + y_i \leqq b_i \text{ for } i = 1,\ldots,n$$

where the b_i are obtainable from the relation

$$b_i = \text{Min } f_i(x) \text{ for } i = 1,\ldots,n$$

thus

$$y_i = \text{Min } f_i(x) - f_i(x) \leqq 0 \text{ for } i = 1,\ldots,n$$

For convenience (III-1) may be rewritten as

$$\begin{aligned} &\text{Minimize} \quad R = -w \cdot y \\ &\text{Subject to } F(x) + y \leq b \end{aligned} \qquad \text{(III-1')}$$

where

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x) \end{bmatrix} \; ; \; y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \; ; \; b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}$$

and

$$w = (w_1, w_2, \ldots, w_n)$$

The function R is a regret function and it is assumed that by minimizing regret, utility is maximized. That is, regret and utility are of the same relation as cost and profit. When making comparisons between alternatives the response obtained by asking "which is preferred?" will be the same as that obtained by asking "which is least dissatisfactory?". Thus it follows that any search which leads to the alternative of minimum regret will simultaneously yield the alternative of highest utility.

To obtain a solution to (III-1) it is advantageous to construct its Lagrangean function $L(\bar{x},u)$, where $\bar{x} = (x,y)$ and u is an n dimensional vector of Lagrange multipliers (dual variables). The function L is given by

$$L(\overline{x},u) = -w \cdot y + u \cdot [F(x) + y - b] \qquad \text{(III-2)}$$

This Lagrangean function will be useful in obtaining an optimal solution to (III-1) for any given w, as will be shown.

If all $f_i(x)$ are assumed to be convex and differentiable the Kuhn-Tucker conditions for optimality are not only necessary but sufficient. For this particular problem the Kuhn-Tucker conditions are

$$\nabla_{\overline{x}} L(\overline{x}^*,u^*) = o \qquad \text{(III-3)}$$

$$u^* \cdot [F(x^*)+y^*-b] = o \qquad \text{(III-4)}$$

$$F(x^*)+y^*-b \leq o \qquad \text{(III-5)}$$

If there exists some $u^* \geq o$ and some x^* and y^* for which (III-3), (III-4) and (III-5) hold then x^* and y^* are global optimal solutions to (III-1) for each given vector of penalty weights w.

Closer examination of equation (III-3) yields some useful results. Equation (III-3) can be rewritten as

$$\frac{\partial L(\overline{x}^*,u^*)}{\partial x} = u^* \cdot F'(x^*) = o \qquad \text{(III-6)}$$

and

$$\frac{\partial L(\bar{x}^*,u^*)}{\partial y} = -w+u^* = o \tag{III-7}$$

where F'(x) is the Jacobian matrix

$$F'(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_m} \end{pmatrix}$$

Equation (III-8) reveals that for any given w the "optimal" set of Lagrange multipliers is known. In a more general setting the task of searching for the value u* would make the usefulness of the first Kuhn-Tucker condition (equation (III-3)) dubious. However, herein lies one of the advantages of the Lagrangean approach. Instead of solving an (m + n) x (m + n) system, it is necessary only to solve the equivalent m x m system of nonlinear equations

$$w \cdot F'(x^*) = o \tag{III-8}$$

Again, in general, this system of equations would be difficult to solve and its difficulty would increase sharply as m increases.

Suppose that the partial of $f_i(x)$ with respect to x_j is some function of only x_j and x_{j+1} if j is odd and of x_j and x_{j-1} if j is even. That is,

$$\frac{\partial f_i(x)}{\partial x_j} = g_{ij}(x_j, x_{j+1}) \text{ for j odd, } i = 1,2,\ldots,n$$

$$\frac{\partial f_i(x)}{\partial x_j} = g_{ij}(x_{j-1}, x_j) \text{ for j even, } i=1,2,\ldots,n$$

then the system of equations (III-8) can be written as

$$\left.\begin{aligned} &\sum_{i=1}^{n} w_i g_{ij}(x_j, x_{j+1}) = o \\ &\sum_{i=1}^{n} w_i g_{ij+1}(x_j, x_{j+1}) = o \end{aligned}\right\} \text{ for } j = 1,3,5,\ldots,m-1 \qquad \text{(III-9)}$$

Equation (III-9) constitutes (m/2) systems of two equations in two unknowns which as a rule are easily solved for values of x_j^*. This is precisely the case in multi-item inventory systems where the x_j and x_{j+1} represent the decision variables for each item in the system.

Each Lagrange multiplier u_i^* is non-zero (since $u_i^* = w_i$ and $w_i \neq o$ for all i). Hence to satisfy complementary slackness (equation (III-4) y^* must be given by

$$y^* = b - F(x^*) \qquad \text{(III-10)}$$

which in turn satisfies primal feasibility (equation (III-5)). Therefore for any particular vector of weights w,x* and y* are global optimal solutions to (III-1).

Interactive Considerations

In using Lagrangean techniques to solve non-linear optimization problems the degree of difficulty increases sharply as n increases. That is, as more multipliers are needed it becomes increasingly difficult to solve for the vector u*. However by using the interactive approach it becomes unnecessary to solve for the value of u* since the decision maker supplies the information needed in the form of the w_i's.

When the GP problem is solved using the method of Lagrange multipliers it becomes unnecessary to solve for u*. If all the w_i's are given one need only solve the m x m system (III-9) for values of x*.

It is assumed here that the decision maker's preference function U is not explicitly known. Translated to goal programming this means that the set of w_i in the objective function R is not precisely known apriori.

Such local information as may be required about the w_i does exist, however, and must be obtained periodically from the decision-maker. Thus an interaction must occur between the mechanism doing the optimization and the decision maker.

To obtain information about the w_i's, an interactive characteristic must be built into the solution procedure. By interacting with the computer through a time sharing program the decision maker can supply any local information about his preference function U which may be useful in determining the w_i's. This procedure is similar, in principle, to that suggested by Dyer [5,6] and Geoffrion [8].

Geoffrion, Dyer and Feinberg [9] state that although there exist some algorithms which might be termed interactive programming algorithms, in most of the approaches in the literature "the interaction is superimposed in an ad hoc manner rather than being dictated by the mathematical programming algorithm itself." In the approach proposed here, it can be shown that the interaction phase is based on considerations derived from the algorithm itself.

The set of w_i's used in the Lagrangean approach to GP are equivalent to those derived by Dyer in [4]. This can be verified in the following manner. Consider the problem statement (III-1'). Suppose that Δx is a perturbation on the vector of decision variables x such that $x^*+\Delta x^* = x'$. Then to maintain complementary slackness, there must be a new value y' such that

$$F(x')+y' = b \qquad \text{(III-11)}$$

where

$$F(x') = F(x^*)+\Delta F(x^*) \tag{III-12}$$

and

$$y' = y^*+\Delta y^* \tag{III-13}$$

That is $F(x')$ and y' are respectively equal to $F(x^*)$ and y^* plus some change $\Delta F(x^*)$ and Δy^*. The changes $\Delta F(x^*)$ and Δy^* are caused by the perturbation Δx^*.

Substitution of (III-12) and (III-13) into (III-11) gives

$$F(x^*)+\Delta F(x^*)+(y^*+\Delta y^*) = b$$

which when regrouped yields

$$\Delta F(x^*) = -\Delta y^* \tag{III-14}$$

After the perturbation of x^* takes place the objective function of (III-1') becomes

$$\begin{aligned} R' &= w\cdot y' \\ &= w\cdot(y^*+\Delta y^*) \\ &= R+w\cdot\Delta y^* \end{aligned}$$

or
$$R'-R = \Delta R = w\cdot\Delta y^* = -w\cdot\Delta F(x^*)$$

thus

$$\frac{\Delta R}{\Delta F(x^*)} = -w \qquad \text{(III-15)}$$

so that for every unit change in $F(x^*)$, the objective function R changes by the amount $-1 \cdot w^t$, where 1 is an n dimensional vector with all components equal to 1.

The limit of (III-10) as $\Delta F(x^*)$ approaches zero is

$$\lim_{\Delta F(x) \to o} \frac{\Delta R}{\Delta F(x^*)} = -w = \frac{\partial R}{\partial F(x^*)}$$

where $(\partial R/\partial F(x^*)$ is a vector of partial derivatives of the form

$$\frac{\partial R}{\partial F(x)} = (\frac{\partial R}{\partial f_1(x)}, \frac{\partial R}{\partial f_2(x)}, \ldots \frac{\partial R}{\partial f_n(x)})$$

If w_1 is chosen to be the reference criteria such that $w_1 = 1$ and all other criteria are given in terms of w_1, then

$$-w_i = (\partial R/\partial f_i(x^*))/(\partial R/\partial f_1(x^*)) \text{ for } i=1,2,\ldots n \qquad \text{(III-16)}$$

which is equivalent to equation (II-6) since (III-1) is a minimization problem (which accounts for the minus sign).

Ijiri [10], suggests that the "...criterion to be considered here is how much increase in a variable (an $f_i(x)$) would be just offset by a unit decrease in some other variable...." The point where the decision maker is indifferent between the vectors $(f_1(x),\ldots,f_n(x))$ and $(f_1(x),\ldots,f_i(x)+\Delta f_i(x)$ $\ldots,f_j(x)-1,\ldots,f_n(x))$, where $f_j(x)$ is an arbitrary reference criterion, is called an indifference point. By obtaining these indifference points it is possible to obtain estimates of the w_i.

A Different Approach to Interaction

For any given value of w, (III-8) may be solved for some optimal decision vector x. The decision vector in turn yields a "consequence" vector F(x).

Suppose one of the w_i's, say w_r, were perturbed. Let w^+ denote the vector containing the perturbed value $w_r+\Delta w_r$, then the system

$$w^+\cdot F'(x) = o$$

can be solved for a new value of x, say $x' = x+\Delta x$ which would in turn yield a new consequence vector F(x'). A comparison can be made between F(x) and F(x') which determines which consequence vector is preferred by the decision maker. The answer to such a comparison would indicate in which direction Δw_r should move. That is, if F(x') is preferred, the new Δw_r should be larger positive if w_r was positive and be larger negative if Δw_r was negative. If F(x) is preferred then desired w_r will be in the interval $[w_r-\Delta w_r,\ w_r+\Delta w_r]$.

This interval can be narrowed as small as desired by further comparisons.

Whenever no further improvement can be obtained by changing w_r then some other w_i, say w_s, is perturbed in a like manner and so on. Once no further improvement can be obtained by changing any of the w_i then the resulting $F(x)$ and its corresponding x are optimal. This procedure is similar in nature to direct search techniques where all variables except one are held constant while the remaining variable is changed until no further improvement is possible, etc.

This procedure searches over values of the dual variables u until some u* and its corresponding x* and y* yield a saddle point of the Lagrangean function, i.e.,

$$L(\bar{x}^*,u^*) \leq L(\bar{x},u^*) \text{ for all feasible } \bar{x} = (x,y)$$

$$L(\bar{x}^*,u^*) \geq L(\bar{x}^*,u) \text{ for all } u \geq o$$

At this point, stepping in any direction on the decision makers regret function increases its value and x*, then, is the most preferred feasible solution.

Interactive Algorithm

The procedure outlined above to obtain a saddle point x*,u* of the Lagrangean function, can be formulated in the forms of an algorithm as follows

Step 1. Determine the target b_i associated with each goal $f_i(x)$. Each b_i must be a number such that it becomes either impossible (i.e., infeasible) or unattractive to achieve an $f_i(x)$ which exceeds the target b_i. At this point the marginal decrease in regret associated with additional gains in $f_i(x)$ is zero (i.e., $\partial R/\partial f_i(x) = 0$) for $f_i(x)$ exceeding b_i. A vector b satisfying the above criteria can be easily found. This is done by extremizing each $f_i(x)$ without regard for any other objectives. Thus the values of b can be determined without any aid from the decision maker.

Step 2. Obtain an initial value for w, w^o. This can be done by an arbitrary assignment or by asking the decision maker to give an initial estimate of relative importance between the criteria.

Step 3. Using the initial value of $w(w^o)$ solve the system of equation (III-5). This yields $F(x)^o$ the initial vector of consequences.

Step 4. Perturb (i.e., increase or decrease) the present value of w say w^k and obtain $F(x)^{k+1}$ and w^{k+1}. If $F(x)^{k+1}$ is the preferred vector of consequences after all possible perturbations of w then terminate[1], otherwise increase k by one and repeat Step 4.

It is important to note that this algorithm coverges to a "global optimal" solution only if the decision maker's preference function U is concave and has no local optima. In

[1]Termination should occur when there exists no perturbation on w which yields a more preferrable value of F(x).

case there exist local optima the algorithm may converge on a local optimum and hence it would not yield the "best compromise" solution.

CHAPTER IV

MODEL AND OBJECTIVES

The model that will be used for illustrative purposes in the remainder of this thesis is the approximate <Q,r>--lost sales model. This is a fixed order quantity (Q) fixed reorder point (r) model. Any multi-item system herein considered will then have all its items stocked under the <Q,r> policy and be in the lost sales case.

An excellent treatment of the approximate <Q,r> model can be found in Hadley and Whitin [10]. They give a detailed explanation of when this model is appropriate, which assumptions are essential for the model to be a good approximation, etc. Nevertheless, it is beneficiary for notational convenience to develop the model here in somewhat less detail.

Assumptions of the Model

The system under consideration must be a transaction reporting system. That is, the inventory level of each item must be examined after every demand on that item. Also, the nature of the demand patterns of all items must be such that the variables Q and r can be considered as continuous.

Other assumptions under which the model is derived are:

(1) All stockouts are considered lost sales.

(2) There exists no substitutions between the items.

(3) The time any item is out of stock can be considered negligible as compared to the total cycle time.

(4) All demand rates are constant over time.

(5) The unit costs of all items are independent of their order quantities.

Since the model will be developed in classical inventory theory terms, one further assumption is needed for the time being. For the discussion that follows all cost factors (ordering, holding and stockout) are assumed known and given. This will allow the model to be later viewed in the context of multiple objective optimization.

Model Development

As previously mentioned, the classical inventory theory approach to optimization is to form a total cost function and optimize it over a set of variables (in this case Q and r). The three components making up this cost function are ordering cost, holding cost, and stockout cost.

Before proceeding some definitions are in order.

Let λ_i = mean yearly demand for item i

μ_i = mean lead time demand for item i

Q_i = order quantity for item i

r_i = reordering point for item i

c_i = cost per unit of item i

π_i = cost per stockout of item i

A_i = ordering cost per order

I = holding cost as a percent (same for all items)

Notice that I has been set equal for all items. This is done since the holding cost is assumed to be a linear function of the item cost c_i.

The average number of orders per year is given by λ_i/Q_i for any item. Therefore the average annual cost of ordering is $\lambda_i A_i/Q_i$. This yields the ordering cost component of the total cost equation. Next the annual cost of holding inventory must be determined. For each item this cost will be Ic_i times the expected number of unit years of stock held per year.

The expected number of units on hand when an order arrives is called the safety stock or buffer stock and is denoted by s_i. Since the order quantity is always Q_i, the expected on hand inventory at the arrival of an order is Q_i+s_i. Each item goes through an inventory circle between the arrivals of two successive orders. Therefore, the expected on hand inventory position varies between Q_i+s_i, at the start of a cycle, and s_i at the end of a cycle. In view of this fact, and the assumption that the mean rate of demand remains constant, the expected on hand inventory decreases linearly from Q_i+s_i to s_i. Thus the average on hand inventory is

$$\frac{(Q_i+s_i)+s_i}{2} = \frac{Q_i}{2} + s_i$$

This quantity is then the average number of unit years of stock held per year.

It is necessary to express the safety stock s_i in terms of the decision variable r_i. Let $\varepsilon(x,r_i)$ be the on hand inventory when an order arrives if the lead time demand is x. Then

$$\varepsilon(x,r_i) = \begin{cases} r_i - x, & r_i - x \geq o \\ o & \text{otherwise} \end{cases}$$

thus the expected on hand inventory when an order arrives (the safety stock, by definition) is

$$s_i = \int_o^\infty \varepsilon(x,r_i)h_i(x)dx = \int_o^{r_i} (r_i - x)h_i(x)dx$$

when $h_i(x)$ is the marginal distribution of lead time demand. The integral can be simplified to a more common form

$$s_i = r_i - \mu_i + \int_{r_i}^\infty xh_i(x)dx - r_iH_i(r_i)$$

where $H_i(r_i)$ = P {lead time demand exceed r_i}. Thus the average annual cost of carrying inventory for each item is

$$Ic_i\left[\frac{Q_i}{2} + r_i - \mu_i + \int_{r_i}^\infty xh_i(x)dx - r_iH_i(r_i)\right]$$

Lastly, the yearly cost of lost sales must be found to complete the total cost function. However this quantity has already been found. It is necessary only to note that the number of lost sales $\eta_i(x,r_i)$ per cycle is given by

$$\eta_i(x,r_i) = \begin{cases} x-r_i, & x-r_i \geq o \\ o & \text{otherwise} \end{cases}$$

Integrating over x gives

$$\eta_i(r_i) = \int_{r_i}^{\infty} (x-r_i)h_i(x)dx$$

which has been evaluated as

$$\eta_i(r_i) = \int_{r_i}^{\infty} xh_i(x)dx - r_iH_i(r_i)$$

The total yearly cost of stockouts (lost sales) is then given by $(\pi_i\lambda_i/Q_i)\eta_i(r_i)$. Having found the three parts in the total cost function for each item all that remains is to put them together.

$$K_i(Q_i,r_i) = \frac{\lambda_iA_i}{Q_i} + Ic_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)] + \frac{\pi_i\lambda_i}{Q_i}\eta_i(r_i)$$

The total yearly average cost for the system is

$$K(\overline{Q},\overline{r}) = \sum_{i=1}^{k}\{\frac{\lambda_iA_i}{Q_i} + Ic_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)] + \frac{\pi_i\lambda_i}{Q_i}\eta_i(r_i)\}$$

where k is the number of items in the system. Here $\overline{Q} = (Q_1, Q_2, \ldots Q_k)$ and $\overline{r} = (r_1, r_2, \ldots, r_k)$ are the vectors of decision variables.

Optimization of the <Q,r> Model

From the above discussion it is evident that $K(\overline{Q},\overline{r})$ is composed of the sum of k objective functions, of the type $K_i(Q_i, r_i)$ that is

$$K(\overline{Q},\overline{r}) = \sum_{i=1}^{k} K_i(Q_i, r_i)$$

If there exists no competition between the items i.e., constraints which involve some or all of the items, then it is a relatively simple process to optimize. In fact, since minimizing cost is the objective here

$$\text{Min } K(\overline{Q},\overline{r}) = \text{Min} \sum_{i=1}^{k} K_i(Q_i, r_i)$$

$$= \text{Min } K_1(Q_1, r_1) + \ldots + \text{Min } K_k(Q_k, r_k)$$

Thus it is necessary to find

$$\text{Min } K_i(Q_i, r_i) = K_i(Q_i^*, r_i^*) \text{ for } i=1,2,\ldots,k$$

If the optimal values Q_i^*, r_i^* satisfy $o < Q_i^* < \infty, o < r_i^* < \infty$, then Q_i^* and r_i^* must be solutions to the equations

$$\frac{\partial K_i}{\partial Q_i} = o = -\frac{\lambda_i \Lambda_i}{Q_i^2} + \frac{Ic_i}{2} - \frac{\pi_i \lambda_i}{Q_i^2} \eta_i(r_i) \tag{IV-1}$$

$$\frac{\partial K_i}{\partial r_i} = o = Ic_i + Ic_i \frac{\partial \eta_i(r_i)}{\partial r_i} + \frac{\pi_i \lambda_i}{Q_i} \eta_i(r_i) \tag{IV-2}$$

Recall that

$$(\partial \eta_i(r_i)/\partial r_i) = \frac{\partial}{\partial r_i}[\int_{r_i}^{\infty} x h_i(x) dx - r_i H_i(r_i)]$$

and

$$\frac{\partial}{\partial a} \int_a^b g(t) dt = -g(a)$$

therefore

$$\frac{\partial \eta_i(r_i)}{\partial r_i} = -r_i h_i(r_i) - \frac{\partial}{\partial r_i}[r_i H_i(r_i)]$$

From the product rule for partial differentiation

$$\frac{\partial}{\partial r_i}[r_i H_i(r_i)] = -r_i h_i(r_i) + H_i(r_i)$$

which finally yields

$$\frac{\partial \eta_i(r_i)}{\partial r_i} = -H_i(r_i)$$

so that

$$\frac{\partial K_i}{\partial r_i} = o = Ic_i - Ic_i H_i(r_i) - \frac{\pi_i \lambda_i}{Q_i} H_i(r_i) \qquad \text{(IV-2')}$$

With the aid of some simple algebra, equations (IV-1) and (IV-2') can be put into the form

$$Q_i = \sqrt{\frac{2\lambda_i [A_i + \pi_i \eta_i (r_i)]}{Ic_i}} \qquad \text{(IV-3)}$$

$$H_i(r_i) = \frac{Q_i Ic_i}{Q_i Ic_i + \pi_i \lambda_i} \qquad \text{(IV-4)}$$

Equations (IV-3) and (IV-4) can be solved numerically for Q_i^* and $H_i(r_i^*)$. By finding the inverse of the cumulative distribution function $H_i(r_i^*)$, r_i^* is obtained. Hence the optimization is complete.

Formulation of Multiple Objectives

It is important to note that the problem considered above was a single objective problem. The objective was to minimize the total yearly variable cost of running an inventory system. Had any of the cost factors A_i, I or π_i, not been available or obtainable, such a straight forward optimization procedure would not have been applicable. This is usually the case in most inventory systems.

To cope with the problems caused by unquantifiable cost factors in inventory control, one might formulate a number of criteria which are used by management to evaluate the performance of any inventory system. The criteria which are commonly used are those which are descriptive of the performance of the system in terms of customer satisfaction, inventory investment, etc. These criteria can also be considered objectives. For example, when management is asked to meet the criteria of lowest investment it is simultaneously having to reach the objective of minimizing inventory investment. So it appears that the words criteria and objective, in this context, are interchangeable and will be used as such.

The Cost Objective

Minimizing cost is one of the most important objectives in designing any system. It is also the criteria most frequently used in evaluation of alternative solutions.

Since it has been assumed that ordering costs and holding costs are the only cost factors which are measurable in dollars the cost objective has the form

$$C(\overline{Q},\overline{r}) = \sum_{i=1}^{k} \{\frac{\lambda_i A_i}{Q_i} + Ic_i[\frac{Q_i}{2} + r_i - \mu_i + \eta_i(r_i)]\}$$

This is simply $K(\overline{Q},\overline{r})$ without the stockout cost term. If $C(\overline{Q},\overline{r})$ were optimized without regard for any other criteria or objectives then it is easily verified that

$$Q_i^* = \sqrt{\frac{2\lambda_i A_i}{Ic_i}}$$

and

$$H_i(r_i) = 1$$

This implies that $r_i^* = o$ and thus the expected number of backorders per cycle would be

$$\eta_i(r_i^*) = \int_o^\infty xh_i(x)dx = \mu_i$$

where μ_i is the mean of the marginal distribution of lead time demand $h_i(x)$. In most cases allowing μ_i stockouts to occur every cycle is not a feasible solution to management. Therefore, other criteria must also be satisfied.

Service Level

The concept of service level is of great importance in optimizing an inventory system. It is usually defined as the ratio of demand satisfied from stock to the total yearly demand. However Hadley and Whitin [10] show that the above definition of service level is equivalent to the expected fraction of time an item is in stock. This, of course, can be expressed as a probability and is denoted by 1-P (out) where P (out) is the probability of being out of stock.

From the definition of service level it is simple to put together a mathematical definition of service level for the <Q,r>--lost sales system.

$$\text{Service level} = 1\text{-}P(\text{out}) = \frac{E(\text{demand satisfied from stock})}{E(\text{yearly demand})}$$

The numerator and denominator of the above expression are already known. They are $\lambda_i-\lambda_i\eta_i(r_i)/Q_i$ and λ_i, respectively. Therefore

$$1\text{-}P(\text{out}) = 1-\eta_i(r_i)/Q_i$$

$$P(\text{out}) = \eta_i(r_i)/Q_i$$

Often P(out) is used in place of service level since a 95 percent service level implies the item is out of stock five percent of the time. As r_i and Q_i increase the service level 1-P(out) increases. However there is a trade-off between the quality of service and the cost $C(\overline{Q},\overline{r})$.

In some instances the manager of the system will place limits on the service levels for certain items. He may wish to keep the probability of being out of stock P(out) from exceeding some level α. Thus his objective is to keep $P(\text{out})_i$ very near the value α_i where α_i may be different from item to item. There are also other items in the system whose service levels are not critical to the operation of the enterprise. The manager, in that case, may wish to use as objectives the service levels of his critical items and the overall system service level.

From the definition of service level the system service level (SSL) is

$$SSL = \frac{\sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} (\lambda_i/Q_i)\eta_i(r_i)}{\sum_{i=1}^{k} \lambda_i} = 1 - \frac{\sum_{i=1}^{k} (\lambda_i/Q_i)\eta_i(r_i)}{\sum_{i=1}^{k} \lambda_i}$$

As before SSL is 1-P (one or more items are out of stock). In fact the system service level reflects the percentage of demands which are satisfied during the year.

Service level can also be defined in terms of dollar volume $c_i\lambda_i$. For the individual items the expression for $P(out)_i$ does not change since c_i cancels from the numerator and the denominator. However the expression for SSL becomes

$$SSL = 1 - \frac{\sum_{i=1}^{k} c_i(\lambda_i/Q_i)\eta_i(r_i)}{\sum_{i=1}^{k} c_i\lambda_i}$$

The number of stockouts per year is sometimes used, in place of or in conjunction with the idea of service level. Although service levels are only a function of the number of stockouts, it is often useful to consider minimization of the number of stockouts independently as another objective.

In the single item case the number of stockouts is given by $\lambda_i \eta_i(r_i)/Q_i$ as was shown earlier. The sum of the individual terms over i yields the total stockouts for the system.

Turn-Over Ratio

Probably the most commonly used criteria in evaluating inventory system performance is the turn-over ratio. Maximizing the turn-over ratio is something most enterprises, which carry inventory, strive for. It is a measure of the volume of business that can be done with any given inventory level.

Turn-over ratio is here defined as the demand satisfied from stock (in dollars) over the average inventory level (in dollars). These two quantities have previously been obtained and so a mathematical statement of the turn-over ratio T is

$$T = \frac{\sum_{i=1}^{k} [c_i\lambda_i - c_i\lambda_i(\eta_i(r_i)/Q_i)]}{\sum_{i=1}^{k} c_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)]}$$

It must be understood that, when used on its own, turn-over ratio may be misleading. Only when used along with other objectives in this section can it be a useful tool in optimization.

Expedites

In many inventory systems whenever a shortage occurs, some person or department is put in charge of expediting any outstanding orders which may exist for that item. If the system is operating in the lost-sales mode then special procurements are made very often. Each special procurement is also considered expedite action. An expedite, then, can be said to occur whenever an item's inventory level reaches a shortage position.

The probability that a stockout occurs within a cycle is $H_i(r_i)$. When this is multiplied by the expected number of cycles per year λ_i/Q_i, it yields the expected number of shortages per year, S_i.

$$S_i = H_i(r_i)\frac{\lambda_i}{Q_i}$$

The total expected expedites per year for the system is then

$$S = \sum_{i=1}^{k} S_i = \sum_{i=1}^{k} H_i(r_i)\frac{\lambda_i}{Q_i}$$

Total expedites per year can be used as an objective to be minimized. When used as one of a multiple set of objectives it can be a valuable criteria for evaluation.

Average Service Level

Whenever there are interdependencies between certain items in an inventory system, it may be of importance to maintain the average service level close to a prespecified target objective. Furthermore it may be desirable that the individual service levels stay close to the target average.

The average service level is defined, in terms of $P(out)_i$, as follows

$$\text{Average S.L.} = 1-\bar{P}(out) = 1-\frac{1}{k}\sum_{i=1}^{k} P(out)_i = 1-\frac{1}{k}\sum_{i=1}^{k} \frac{\eta_i(r_i)}{Q_i}$$

For a measure of dispersion about the average it would appear suitable to use the unbiased estimator of sample deviation. It is given by

$$\text{Deviation from the average} = \sqrt{\frac{\sum(P(out)_i-\bar{P}(out))^2}{k-1}}$$

These two objectives insure the manager that no unduly large differences in service levels will exist after optimization. They can be used to restrict the range of service levels for all the items in the system or for different subsets within the system.

It is important to reiterate that the individual objectives presented in this chapter seldom have real value on their own. Only when there exists a strong and binding constraint on the system and that constraint is represented by one of these objectives is it suggested that they be used individually.

Objectives which deal with the number of orders placed or the storage space availability are sometimes used. However, they are not included here since ordering costs and holding costs are assumed to be known. Nevertheless, if there are constraints on the system which are violated by the solution obtained using those objectives mentioned in this chapter, then the extra constraints must be introduced and a new feasible solution obtained.

It is also necessary to point out that although the <Q,r>--lost sales model has been utilized here, the same objectives can be derived for most of the inventory models in the literature.

CHAPTER V

APPLICATION OF THE LAGRANGEAN APPROACH

The characteristics of the model developed in the previous chapter are such that is is well suited for solution with a procedure such as the Generalized Lagrangean Multiplier Technique. Since the model has been formulated as a multiple criteria model, Interactive Goal Programming can be effectively used as an optimization tool. It is important to note that the objective constraints used here are of the form convex $\leq$ or concave $\geq$.[1]

A Two Dimensional Inventory Problem

To demonstrate more clearly some of the features of the Lagrangean approach to Interactive Goal Programming a simple two dimensional problem is considered here. The dimension of the problem is equal to the number of goals which are under consideration.

There are two variables which describe the operating doctrine for any item in a <Q,r> system, Q and r. Thus a system containing k items has 2k decision variables. However in an interactive optimization mode these variables are never directly considered by the decision maker. In fact the decision maker sees only information which is the consequence of any change in the decision variables.

[1]See Hadley and Whitin [10, Ch. 4] for proof of convexity or concavity of the $f_i(x)$.

Problem Formulation

Consider a <Q,r>--lost sales inventory system. Suppose that the decision maker has chosen two goals to optimize. Further suppose that he has selected ordering plus holding cost to be one of his objectives and system service level as another. In this case, the tradeoff will be dollars vs. customer satisfaction.

Again the general formulation is

$$\text{Minimize} \quad R = \sum_{i=1}^{n} w_i y_i$$

$$\text{s.t.} \quad f_i(x)+y_i \lesseqqgtr b_i \ \text{ for } i=1,2,\ldots,n$$

In this particular case $n=2$, $x = (\bar{Q},\bar{r}) = (Q_1,\ldots,Q_k,r_1,\ldots,r_k)$ and

$$f_1(x) = \sum_{i=1}^{k} \frac{\lambda_i A_i}{Q_i} + \sum_{i=1}^{k} Ic_i\left[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)\right]$$

$$f_2(x) = 1-\left(\sum_{i=1}^{k}\lambda_i\right)^{-1}\sum_{i=1}^{k}(\lambda_i/Q_i)\eta_i(r_i)$$

So that the problem reads

$$\text{Minimize} \quad R = -w_1y_1+w_2y_2 \tag{V-1a}$$

$$\text{Subject to} \quad \sum_{i=1}^{k}\frac{\lambda_i A_i}{Q_i} + \sum_{i=1}^{k} Ic_i\left[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)\right]+y_1 \le b_1 \tag{V-1b}$$

and

$$1-\left(\sum_{i=1}^{k}\lambda_i\right)^{-1}\sum_{i=1}^{k}(\lambda_i/Q_i)\eta_i(r_i)+y_2\geq b_2 \qquad \text{(V-1c)}$$

The goals b_i must be such that they are simultaneously unattainable.[1] They can be found by optimizing over each objective without regard to any others. In the case of cost, the goal b_1 is given by

$$b_1 = \sum_{i=1}^{k}\frac{\lambda_i A_i}{Q_i} + \sum_{i=1}^{k} Ic_i\left[\frac{Q_i^o}{2}+r_i^o-\mu_i+\eta_i(r_i^o)\right] \qquad \text{(V-2)}$$

where

$$Q_i^o = \sqrt{\frac{2\lambda_i A_i}{Ic_i}} \quad \text{for } i=1,2,\ldots,k$$

and

$$r_i^o = o \qquad \text{for } i = 1,2,\ldots,k$$

The best possible value for service level is clearly 1.0 when service level is expressed as the probability that an order can be served from stock.

The Lagrangean of (V-1) is

---

[1]See Chapter III.

$$L(x,u) = \sum_{i=1}^{n} w_i y_i + u_1\{\sum_{i=1}^{k} \frac{\lambda_i A_i}{Q_i} + Ic_i[\frac{Q_i}{2} + r_i - \mu_i + \eta_i(r_i)] + y_1 - b_1\}$$

$$+u_2\{1-(\sum_{i=1}^{k}\lambda_i)^{-1}\sum_{i=1}^{k}(\lambda_i/Q_i)\eta_i(r_i)+y_2-b_2\}$$

where $x = (\overline{Q},\overline{r},y)$ is the 2k+2 vector of decision variables. In order to obtain the saddle point of L, x* and u*, it is necessary to find a solution to

$$\nabla_x L(x^*,u^*) = o \tag{V-3}$$

Taking partials as indicated by (V-3) and (V-4) yields

$$\frac{\partial L}{\partial Q_i} = -\frac{u_1^*\lambda_i A_i}{Q_i^2} + \frac{u_1^* Ic_i}{2} + u_2^*(\sum_{i=1}^{k}\lambda_i)^{-1}(\lambda_i/Q_i^2)\eta_i(r_i) = o \tag{V-4}$$

$$\frac{\partial L}{\partial r_i} = u_1^* Ic_i - u_1^* Ic_i H_i(r_i) + u_2^*(\sum_{i=1}^{k}\lambda_i)^{-1}(\lambda_i/Q_i)H_i(r_i) = o \tag{V-5}$$

$$\frac{\partial L}{\partial y_i} = w_i + u_i^* = o \tag{V-6}$$

When (V-4) through (V-6) are solved for Q_i, r_i, y_i and u_i^* the result is the following set of equations

$$Q_i = \sqrt{\frac{2\lambda_i[A_i-(u_2^*/u_2^*)(\sum_{i=1}^{k}\lambda_i)^{-1}\eta_i(r_i)]}{Ic_i}} \quad \text{for } i=1,\ldots,k \tag{V-7}$$

$$H_i(r_i) = \frac{Q_i Ic_i}{Q_i Ic_i-(u_2^*/u_1^*)(\sum_{i=1}^{k}\lambda_i)^{-1}\lambda_i} \quad \text{for } i=1,\ldots k \tag{V-8}$$

$$w_1 = u_1^*;\ u_1^* \geq o \tag{V-9}$$

$$w_2 = -u_2^*;\ u_2^* \leq o \tag{V-10}$$

Since $w \neq o$, complementary slackness can only be satisfied if

$$y_1^* = b_1-\sum^{k}\{\frac{\lambda_i A_i}{Q_i^*}+Ic_i^*[\frac{Q_i^*}{2}+r_i^*-\mu_i+\eta_i(r_i^*)]\} \tag{V-11}$$

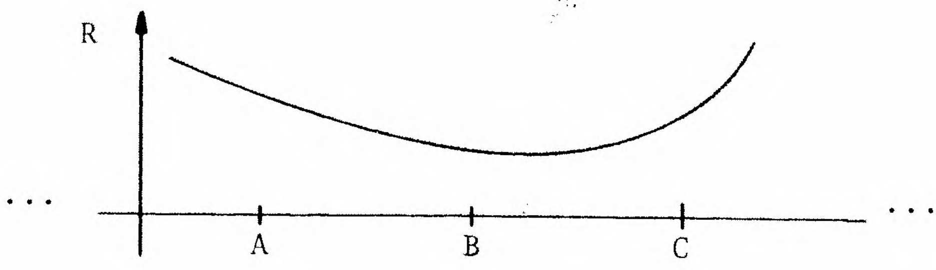
$$y_2^* = b_2-[1-(\sum_{i=1}^{k}\lambda_i)^{-1}\sum_{i=1}^{k}(\lambda_i/Q_i^*)\eta_i(r_i^*)] \tag{V-12}$$

where Q_i^* and r_i^* are solutions to (V-7) and (V-8).

When equations (V-7) and (V-8) are compared to equations (IV-3) and (IV-4) an interesting observation can be made. Notice that the factor $(u_2/u_1)(\sum\lambda_i)^{-1}$ is everywhere the equivalent of π_i. This brings out an implicit assumption which is made when the decision maker uses system service level as his only other goal besides cost. Clearly, the assumption is that the stockout penalty cost is the same for all the items in the system. However, for this simple illustrative example, such technicalities can be dispensed with.

The solution procedure will be as follows:

1. Replace u_1 and u_2 with $-w_1$ and w_2 in equations (V-7) and (V-8).
2. Obtain initial estimates of w_1 and w_2 from the decision maker.
3. Calculate or set values for b_1 and b_2.
4. Obtain a solution to (V-7) and (V-8) by any appropriate method (fixed point iteration, Newton's Method, etc.) and calculate $F(x)^o$.
5. Change one of the w's, say w_2, and obtain a new F(x), say F(x'), by the same procedure as in Step 3.
6. Compare F(x) to F(x'). Continue to change w_2 until by repeated comparisons of $F(x)^k$ and $F(x)^{k+1}$ it can be determined that the desired F(x) is within an interval, say $[w_{2L}, w_{2R}]$.
   The interval $[w_{2L}, w_{2R}]$ is determined as in the figure below. Suppose F(x) at B is preferred over F(x) at A. Then w_2 is moved in the direction of the preference to point C. If F(x) at B is preferred over F(x) at C then the desired F(x) must lie in the interval [A,C].



This, of course, assumes that R (the regret function) does not have any local optima.

7. Once an interval has been located it can be narrowed by further comparisons between consequence vectors within the interval. This can be accomplished by a method similar to Fibonacci search, Golden Sections or Interval Bisection.

A Numerical Example

The algorithm above was programmed in Fortran on a Univac 1108 computer in order to illustrate the tradeoffs which must be made by a decision maker when using this type of problem solving approach. It also serves to illustrate the interdependencies between the F(x)'s and w's.

For the numerical examples it will be assumed for the sake of conciseness, that the lead time demand distribution functions h_i are normally distributed with mean μ_i and variance σ_i^2. So that

$$h_i(r_i) = \frac{1}{\sigma_i\sqrt{2\pi}}e^{[-1/2(\frac{r_i-\mu_i}{\sigma_i})^2]}$$

and

$$H_i(r_i) = \int_{r_i}^{\infty} h_i(x)dx$$

Hence, the expected number of stockouts per cycle is given by

$$\eta_i(r_i) = (\mu_i - r_i)H_i(r_i) + \sigma_i h_i(r_i)$$

where

$$r_i = t\sigma_i + \mu_i$$

The variable t is obtained by finding the inverse of the cumulative distribution function H_i. Thus if

$$\alpha = \frac{Q_i Ic_i}{Q_i Ic_i + \frac{w_2}{w_1}(\sum\lambda_i)^{-1}\lambda_i}$$

then

$$t = H^{-1}(\alpha)$$

For the case of the normal distribution, t can be found from tables of the cumulative normal distribution function.

For this numerical example suppose there are five items in the system and their characteristics are given in the table below.

Item No.	λ	μ	σ	c	A
1	10000	700	100	20	50
2	100000	500	150	25	100
3	55000	5000	200	35	75
4	11000	100	20	500	400
5	20000	1000	175	100	200

Let I = .25 be used as a carrying charge per dollar of inventory per year. It will be the same for all five items.

The input to the program consists of:

1. Subjective estimates of w_1 and w_2
2. The number of items in the system
3. The number of goals
4. The carrying charge (I)
5. The table of characteristics.

From this data the value of b_1 is calculated using equation (V-2). b_2 is set equal to 1.0. Next two vectors F(x) and F(x') are computed using values of x which are solutions to (V-7) and (V-8). A computer printout demonstrating Steps 5 and 6 of the algorithm is shown. Vectors A and B represent F(x) and F(x'), respectively.

ITEM NØ.	ØRDER QUANTITY	REØRDER PØINT	STØCKØUTS PER YEAR	SERVICE LEVEL	ØRD.+HØLD. CØST
1	491.25	717.79	644.15	.9356	2493.09
2	1857.33	595.36	1286.46	.9871	11933.59
3	1062.88	5081.77	2355.53	.9572	9644.85
4	268.23	74.00	1103.62	.8997	33282.32
5	621.34	906.35	4070.09	.7965	15024.22
			------------		------------
			9459.87		72378.07

****

	A	B
CØST	72378.07	74169.89
SYSTEM SERVICE LEVEL	.95174	.97161
CØRRESPØNDING W2	64000.00	128000.00

```
**************************
* DØ YØU PREFER A ØR B ? *
**************************
```

A

	A	B
CØST	72378.07	71082.57
SYSTEM SERVICE LEVEL	.95174	.92318
CØRRESPØNDING W2	64000.00	32000.00

```
**************************
* DØ YØU PREFER A ØR B ? *
**************************
```

A

THE SØLUTIØN LIES BETWEEN 32000.0 AND 128000.0

```
*********************************
* DØ YØU DESIRE TØ CØNTINUE ? *
*********************************
```

YES

	A	B
CØST	72378.07	73365.28
SYSTEM SERVICE LEVEL	.95174	.96434
CØRRESPØNDING W2	64000.00	96000.00

```
*************************
* DØ YØU PREFER A ØR B ? *
*************************
```

B

THE SØLUTIØN LIES BETWEEN 64000.0 AND 128000.0

```
******************************
* DØ YØU DESIRE TØ CØNTINUE ? *
******************************
```

NØ

ITEM NØ.	ØRDER QUANTITY	REØRDER PØINT	STØCKØUTS PER YEAR	SERVICE LEVEL	ØRD.+HØLD. CØST
1	493.45	742.55	449.85	.9550	2570.63
2	1857.18	630.74	855.83	.9914	12104.63
3	1064.81	5130.41	1601.85	.9709	9944.91
4	269.07	78.23	947.60	.9139	33345.31
5	630.41	948.18	3134.29	.8433	15399.54
			6989.43		73365.02

NØRMAL EXIT. EXECUTIØN TIME: 437 MLSEC.

Extension Into Four Dimensions

Consider again a <Q,r>--lost sales inventory system composed of k items. Now in addition to the two goals used earlier, suppose management wants to use turn-over rate and expedites as additional goals for optimization. This fact complicates the tradeoffs to be made. Now, not only need tradeoffs be made between dollars and customer satisfaction, they must be made pairwise between all four goals.

When using turn-over rate as an objective certain difficulties arise. The form of the turn-over rate goal when expressed in the standard form should be

$$\frac{\sum_{i=1}^{k} [c_i\lambda_i - c_i(\lambda_i/Q_i)\eta_i(r_i)]}{\sum_{i=1}^{k} c_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)]} + y_3 \geq b_3$$

if turnover rate is defined to be the third goal. When a partial derivative of the above expression is taken with respect to any Q_i or r_i it can be seen (from the quotient rule) that the result will contain all variables Q_i and r_i. Hence simple equations of the form (V-7) and (V-8) cannot be obtained, making the solution of the system $u^* F'(x) = o$ an ardous task.

In order to obtain k separable sub-systems similar to (V-7) and (V-8) it is necessary to make a simplifying adjustment. The adjustment is to redefine turn-over rate to be T times the average inventory level, or

$$\sum_{i=1}^{k}[c_i\lambda_i-c_i(\lambda_i/Q_i)\eta_i(r_i)] = T\sum_{i=1}^{k}c_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)]$$

Thus the goal equation for turn-over rate is rewritten as

$$\sum_{i=1}^{k}[c_i\lambda_i-c_i(\lambda_i/Q_i)\eta_i(r_i)]+y_3\geq b_3\sum_{i=1}^{k}c_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)]$$

The summations on both sides allow partials with respect to any Q_i or r_i, say Q_s and r_s, to contain only the variable Q_s and r_s. This makes possible the same kind of separability obtained in the two dimensioned example. This type of complication may occur with other measures defined earlier, however a similar approach to that used on turn-over rate may be used for simplifying purposes.

The formulation for the four goal problem is then

$$\text{Minimize } -w_1y_1+w_2y_2+w_3y_3-w_4y_4 \tag{V-13a}$$

$$\text{Subject to } \sum_{i=1}^{k}\frac{\lambda_iA_i}{Q_i}+\sum_{i=1}^{k}Ic_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)]+y_1\leq b_1 \tag{V-13b}$$

$$1-(\sum_{i=1}^{k}\lambda_i)^{-1}\sum_{i=1}^{k}(\lambda_i/Q_i)\eta_i(r_i)+y_2\geq b_2 \tag{V-13c}$$

$$\sum_{i=1}^{k}[c_i\lambda_i-c_i(\lambda_i/Q_i)\eta_i(r_i)]+y_2\geq b_3\sum_{i=1}^{k}c_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)] \tag{V-13d}$$

$$\sum_{i=1}^{k} H_i(r_i)(\lambda_i/Q_i)+y_4 \leq b_4 \tag{V-13e}$$

The goal targets b_1 and b_2 are obtained as before. b_3 is set arbitrarily large and b_4 is obviously zero since zero expedites is the best that can be expected. Since y_1 and y_4 will always be negative, w_1 and w_4 must be negative, otherwise the objective function would improve (decrease) with a worsening of the deviations from goals 1 and 4.

The Lagrangean formulation of (V-13) is

$$L(x,u) = \sum_{i=1}^{n} w_i y_i + u_1\{\sum_{i=1}^{k} \frac{\lambda_i A_i}{Q_i} + \sum_{i=1}^{k} Ic_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)]+y_1-b_1\}$$

$$+u_2\{1-(\sum_{i=1}^{k}\lambda_i)^{-1}\sum_{i=1}^{k}(\lambda_i/Q_i)\eta_i(r_i)+y_2-b_2\}$$

$$+u_3\{\sum_{i=1}^{k}[c_i\lambda_i-c_i(\lambda_i/Q_i)\eta_i(r_i)+y_3-b_3\sum_{i=1}^{k}c_i[\frac{Q_i}{2}+r_i-\mu_i+\eta_i(r_i)]]\}$$

$$+u_4\{\sum_{i=1}^{k}H_i(r_i)(\lambda_i/Q_i)\eta_i(r_i)+y_4\}$$

The saddle point of L(x,u) is again obtained by taking partials as indicated by equations (V-3) and (V-4). Solving the resulting equations for Q_i and $H_i(r_i)$ yields:

$$Q_i = \left\{\frac{2\lambda_i[\Lambda_i+(u_4/u_1)H_i(r_i)-((u_2/u_1)(\sum_{i=1}^{k}\lambda_i)^{-1}+(u_3/u_1)c_i)\eta_i(r_i)]}{c_i(I-(u_3/u_1)b_3)}\right\}^{1/2} \quad \text{(V-14)}$$

$$H_i(r_i) = \frac{[(u_3/u_1)b_3c_i-Ic_i]Q_i+(u_4/u_1)\lambda_i h_i(r_i)}{[(u_2/u_1)\lambda_i+(u_3/u_1)c_i\lambda_i\sum_{i=1}^{k}\lambda_i+(u_3/u_1)b_3c_i(\sum_{i=1}^{k}\lambda_i)]\sum_{i=1}^{k}\lambda_i} \quad \text{(V-15)}$$

$$w_1 = u_1^*;\ u_1 \geq o \quad \text{(V-16)}$$

$$w_2 = -u_2^*;\ u_2 \leq o \quad \text{(V-17)}$$

$$w_3 = -u_3^*;\ u_3 \leq o \quad \text{(V-18)}$$

$$w_4 = u_4^*;\ u_4 \geq o \quad \text{(V-19)}$$

To satisfy complementary slackness

$$y_1^* = b_1 - \sum_{i=1}^{k}\frac{\lambda_i A_i}{Q_i^*} - \sum_{i=1}^{k} Ic_i\left[\frac{Q_i^*}{2}+r_i^*-\mu_i+\eta_i(r_i^*)\right] \quad \text{(V-20)}$$

$$y_2^* = b_2 - [1-(\sum_{i=1}^{k}\lambda_i)^{-1}\sum_{i=1}^{k}(\lambda_i/Q_i^*)\eta_i(r_i^*)] \quad \text{(V-21)}$$

$$y_3^* = b_3 \sum_{i=1}^{k} c_i\left[\frac{Q_i^*}{2}+r_i^*-\mu_i+\eta_i(r_i^*)\right] - \sum_{i=1}^{k}\left[c_i\lambda_i-c_i(\lambda_i/Q_i^*)\eta_i(r_i^*)\right] \qquad \text{(V-22)}$$

$$y_4^* = b_4 - \sum_{i=1}^{k} H_i(r_i^*)(\lambda_i/Q_i^*) \qquad \text{(V-23)}$$

where Q_i^* and r_i^* are solutions to (V-14) and (V-15).

Solving equations (V-14) and (V-15) for Q_i^* and r_i^* is somewhat more difficult than solving (V-7) and (V-8). This is so because the term $h_i(r_i)$ appears in equation (V-15) which must now be solved using a numerical method such as Fibonacci search. Aside from this fact, however, the solution procedure for the four goal problem varies from that used in the two goal problem only in that there are now three w's to be perturbed (one stays constant as the reference criterion).

CHAPTER VI

CONCLUSIONS

The area of inventory systems optimization is important, both from the mathematical and profitability points of view. While modeling inventory systems can become quite elegant mathematically there is also profit to be made from designing efficient inventory systems. Nevertheless, it is rare to see the two aspects come together.

The same is perhaps true of many other areas of Operations Research. Extensive mathematical models exist to deal with numerous problem situations, yet in the business world they are seldom used. Reasons for this can be attributed to many factors but surely, the fact that businessmen do not fully trust the so-called optimum solutions arrived at by "magical-mystical" means, must be at the top of the list. The operations research practitioner has heretofore been guilty of asking management for undeterminable cost parameter estimates and then showing the "optimal" solution back to the manager. Even when the solution procedures are reasonably well explained, and properly packaged and sold to management, most managers would rather trust their intuition.

The approach proposed in this thesis is a positive step in the direction of overcoming some of the problems often encountered in application of O.R. principles. By using multiple objectives and an interactive approach the languages of business and optimization can be brought closer together. This is so because no longer are solutions brought in on a silver platter to be accepted or rejected, rather at each step of the way it is the communication between the decision maker and the optimization device which brings about an acceptable solution.

In short, the "multi-objective/interactive" approach to optimization recognizes the decision-maker's ability to make decisions, which is, after all, his "raison de etre." Also this approach has a distinct psychological advantage in that it is difficult for a decision maker to reject a solution which he himself created.

Advantages of the Proposed Interactive Method

In evaluating the proposed solution procedure to the multi-objective problem it is advantageous to compare it with the interactive goal programming algorithm of Dyer [5]. Although the comparison can only be made on subjective and theoretical grounds, it is possible to assess the improvements which have been made over the Dyer method.

In a basic sense, the two methods are designed to solve the general goal programming problem. However, they differ in approach. Dyer uses a primal approach to the problem particularly the Frank-Wolfe algorithm for non-linear programming with some modifications. The approach proposed here uses the Lagrangean formulation, or dual of the problem, to obtain a solution. Whenever the system of equations $u \cdot F'(x) = o$ has properties which make it particularly simple to solve, the Lagrangean approach should be desirable since it always yields a system of this type. Problems such as multi-item inventory systems which may contain thousands of decision variables, make it particularly useful to adopt this technique for solution. This is so because the system $u \cdot F'(x) = o$ is separable into small two-variable subsystems which can be readily solved.

The approach proposed in this thesis bypasses the need for two types of interactions with the decision maker. That is, rather than first determine w's by an extensive set of questions and answers and then determining an approximation to a step size problem, as in Dyer's method, the method proposed here goes from point to point on the decision maker's preference (or regret) function using information received from the decision maker as a guide.

In using the Lagrangean approach it is unnecessary to independently determine an initial feasible point, as is required in Step 2 of Dyer's algorithm. All that is required to get the proposed algorithm started is a subjective estimate of the importance factors w_i.

Areas of Further Research

Although there seem to be several subjective reasons which suggest that the method proposed here should be operationally superior (where applicable) to that of Dyer, there is a need for a head-to-head comparison on a controlled experiment basis. Furthermore, the same type of comparison can be made to other approaches in the literature. However, this is beyond the scope of this thesis.

The Lagrangean approach assumes that certain assumptions hold on the objective function as well as on the constraints. For a minimization problem these are, convexity, continuity, and differentiability. For problems not adhering to those assumptions the solution obtained by a Lagrange multiplier technique cannot be guaranteed to be a minimum. Therefore it would be of interest to study the applicability of the proposed approach to problems which do not totally adhere to the necessary assumptions. A further topic for research along these lines is to prove convergence of the algorithm and to define conditions under which convergence can be achieved.

This research has demonstrated the applicability of the Lagrangean approach to multi-objective optimization in the area of multi-item inventory systems. It remains to be shown whether it is applicable to some or all of the inventory models currently known. Other areas of applicability can also be studied.

Within the algorithm itself several questions remain open. First, the method for perturbation of the w's may be further defined. That is, the question of how much to perturb each w_i to achieve most rapid convergence must be answered. It is also possible to consider the effect of errors made by the decision maker when choosing between alternatives. It may in fact be necessary to introduce a capability for returning to a previous point in the solution space if after an error in judgement the decision maker decides that the direction he is taking is not to his liking.

APPENDIX A

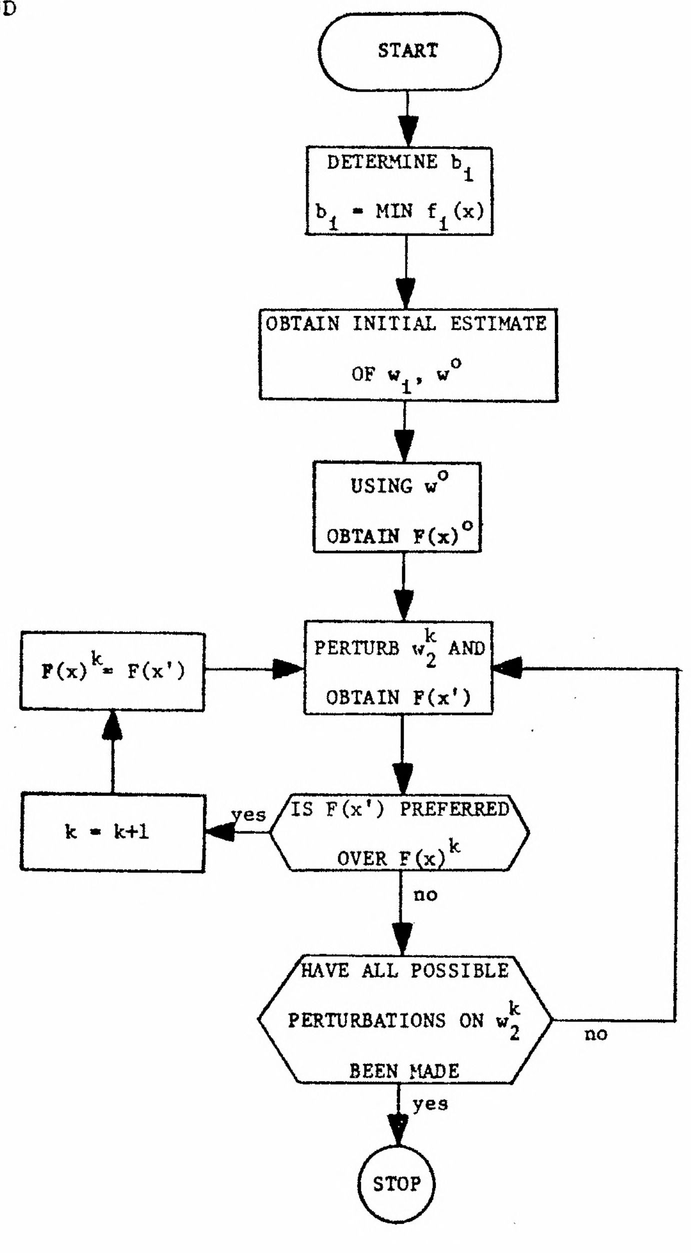
FORTRAN PROGRAM FOR TWO DIMENSIONAL EXAMPLE

```
      DIMENSIØN FWTEST(10),FWR(10),FWL(10),Y(10),F(10)
      CØMMØN XLAMDA(81),XMU(81),C(81),Q(81),SIGMA(81),R(81),EHTA(81)
     *,ALPHA(81),B(10),CI,N,L,A(81),W1
001   FØRMAT( )
      READ(5,001)W1,W
      READ(5,001)L,N,CI
      DØ 009 I=1,L
      READ(5,001)XLAMDA(I),XMU(I),SIGMA(I),C(I),A(I)
      Q(I)=SQRT(2*XLAMDA(I)*A(I)/(CI*C(I)))
      B(1)=B(1)+XLANDA(I)*A(I)/Q(I)+CI*C(I)*Q(I)/2
      SUMLAM=SUMLAM+XLAMDA(I)
009   CØNTINUE
      B(2)=1.0
      CALL FQR(W,F,Y)
      WRITE(6,180)
      DØ 050 I=1,L
      SL=1-EHTA(I)/Q(I)
      BØ=(1-SL)*XLAMDA(I)
      BØTØT=BØTØT+BØ
      CØST=XLAMDA(I)*A(I)/Q(I)+CI*C(I)*(Q(I)/2+R(I)-
     *XMU(I)+EHTA(I))
      CØSTT=CØSTT+CØST
      WRITE(6,170)I,Q(I),R(I),BØ,SL,CØST
050   CØNTINUE
      WRITE(6,190)
      WRITE(6,195)BØTØT,CØSTT
      WRITE(6,200)
      WR=2*W
      CALL FQR(WR,FWR,Y)
      CALL CMPARE(F,FWR,W,WR,ANS)
      IF(ANS.EQ.'A') GØ TØ 005
      GØ TØ 010
C ************ DECREASE W2 ***********
005   WL=.5*W
      CALL FQR(WL,FWL,Y)
      CALL CMPARE(F,FWL,W,WL,ANS)
      IF(ANS.EQ.'A') GØ TØ 020
      WR=W
      W=WL
      DØ 15 I=1,N
      FWR(I)=F(I)
      F(I)=FWL(I)
015   CØNTINUE
      GØ TØ 005
C *********** INCREASE W2 ***********
010   WL=W
      W=WR
      DØ 25 I=1,N
      FWL(I)=F(I)
      F(I)=FWR(I)
025   CØNTINUE
      WR=2*W
      CALL FQR(WR,FWR,Y)
      CALL CMPARE(F,FWR,W,WR,ANS)
```

```
      IF(ANS.EQ.'A') GØ TØ 020
      GØ TØ 010
C    * ANSWER IS WITHIN AN INTERVAL (WL,WR) *
020   CØNTINUE
      CALL INFØ(WL,WR)
150   FØRMAT(A3)
160   FØRMAT(' *******************************'/' * DØ YØU DESIRE TØ CØ
     *TINUE ? *'/' *******************************'/)
      WRITE(6,160)
      READ(5,150)GØNØGØ
      IF(GØNØGØ.NE.'YES')GØ TØ 400
      DW=(WR-W)/2
      WTEST=W+DW
      CALL FQR(WTEST,FWTEST,Y)
      CALL CMPARE(F,FWTEST,W,WTEST,ANS)
      IF(ANS.EQ.'A') GØ TØ 100
C ************ INCREASE W2 *************
      WL=W
      W=WTEST
      DØ 35 I=1,N
      FWL(I)=F(I)
      F(I)=FWTEST(I)
035   CØNTINUE
      GØ TØ 020
C ************ DECREASE W2 **************
100   WR=WTEST
      DW=(W-WL)/2
      WTEST=W-DW
      CALL FQR(WTEST,FWTEST,Y)
      CALL CMPARE(F,FWTEST,W,WTEST,ANS)
      IF(ANS.EQ.'A') GØ TØ 300
C    * CLØSE THE INTERVAL FRØM THE RIGHT *
      WR=W
      W=WTEST
      DØ 045 I=1,N
      FWR(I)=F(I)
      F(I)=FWTEST(I)
045   CØNTINUE
      GØ TØ 020
C    * CLØSE THE INTERVAL FRØM THE LEFT *
300   WL=WTEST
      DØ 055 I=1,N
      FWL(I)=FWTEST(I)
055   CØNTINUE
      GØ TØ 020
400   CALL FQR(W,FW,Y)
      WRITE(6,180)
       BØTØT=0
       CØSTT=0
      DØ 500 I=1,L
      SL=1-EHTA(I)/Q(I)
      BØ=(1-SL)*XLAMDA(I)
      BØTØT=BØTØT+BØ
      CØST=XLAMDA(I)*A(I)/Q(I)+CI*C(I)*(Q(I)/2+R(I)-
```

```
     *XMU(I)+EHTA(I))
      CØSTT=CØSTT+CØST
      WRITE(6,170)I,Q(I),R(I),BØ,SL,CØST
170   FØRMAT(/3X,I3,3X,3(F11.2,1X),4X,F5.4,1X,F11.2)
180   FØRMAT(//,2X,'ITEM',6X,'ØRDER',6X,'REØRDER',4X,'STØCKØUTS',6X,
     *'SERVICE'1X,'ØRD.+HØLD.'/,3X,'NØ.',5X,'QUANTITY',5X,'PØINT'
     *,6X,'PER YEAR',7X,'LEVEL',4X,'CØST'/)
190   FØRMAT(33X,'-----------',11X,'-----------')
195   FØRMAT(33X,F11.2,11X,F11.2/)
200   FØRMAT(' ***********************************************************
     *********'/)
500   CØNTINUE
      WRITE(6,190)
      WRITE(6,195)BØTØT,CØSTT
      STØP
C********       END ØF MAIN PRØGRAM        ***********
C
C********    SUBRØUTINE CØMPARE     ***********
      SUBRØUTINE CMPARE(F,FWN,W,WN,ANS)
      CØMMØN DUM(659),N
      DIMENSIØN FWN(10),F(10)
002   FØRMAT(' **************************'/' * DØ YØU PREFER A ØR B ? *
     */' **************************'/)
      WRITE(6,005)
      WRITE(6,003)F(1),FWN(1),F(2),FWN(2),W,WN
      WRITE(6,002)
      READ(5,004)ANS
003   FØRMAT(' CØST',16X,2(F15.2,1X)//' SYSTEM SERVICE LEVEL'
     *,9X,F6.5,10X,F6.5//' CØRRESPØNDING W2',4X,2(F15.2,1X)/)
004   FØRMAT(A3)
005   FØRMAT(//,29X,'A',15X,'B'/)
      RETURN
C********    SUBRØUTINE INFØ    ***********
      SUBRØUTINE INFØ(WL,WR)
040   FØRMAT(//' THE SØLUTIØN LIES BETWEEN 'F12.1,3X,'AND',F12.1/)
      WRITE(6,040)WL,WR
      RETURN
C********       SUBRØUTINE FQR       ***********
      SUBRØUTINE FQR(W,FO,Y)
      DIMENSIØN FO(10),Y(10)
      CØMMØN XLAMDA(81),XMU(81),C(81),Q(81),SIGMA(81),R(81),EHTA(81)
     *,ALPHA(81),B(10),CI,N,L,A(81),W1
      I=1
      FO(1)=0.0
      SUMSØ=0.0
      FACTØR=(W/W1)/SUMLAM
010   QIC=CI*C(I)*Q(I)
      ALPHA(I)=QIC/(QIC+FACTØR*XLAMDA(I))
      X=TINØRM(1-ALPHA(I),$025)
025   R(I)=X*SIGMA(I)+XMU(I)
      G=EXP(-.5*X**2)/SQRT(2.0*3.14)
      EHTA(I)=(XMU(I)-R(I))*ALPHA(I)+SIGMA(I)*G
      QTEST=Q(I)
      Q(I)=SQRT(2*XLAMDA(I)*(A(I)+FACTØR*EHTA(I))/(CI*C(I)))
```

```
      IF(ABS(QTEST-Q(I)).GT..5) GØ TØ 010
      IF(I.EQ.L) GØ TØ 020
      I=I+1
      GØ TØ 010
020   DØ 40 I=1,L
      FO(1)=FO(1)+XLAMDA(I)*A(I)/Q(I)+CI*C(I)*(Q(I)/2+R(I)-
     *XMU(I)+EHTA(I))
      SUMSØ=SUMSØ+XLAMDA(I)*EHTA(I)/Q(I)
040   CØNTINUE
      FO(2)=1-(1/SUMLAM)*SUMSØ
      Y(1)=B(1)-FO(1)
      Y(2)=B(2)-FO(2)
      ØBJ=-W1*Y(1)+W*Y(2)
      RETURN
      END
```



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